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Interaction of a warm plasma column with high-frequency electric fields: purely growing parametric instability

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Abstract. A formalism for the derivation of the dispersion relation for potential surface waves which can propagate on a bounded warm plasma (plasma column) immersed in a high-frequency electric field $\boldsymbol{E} = \boldsymbol{E}_0 \sin \omega_0 t$ is described. The dispersion relation is examined for the case of strong and weak external electric fields. The threshold field and the growth rate of the purely growing surface parametric instability in a warm plasma column are obtained. It is shown that the threshold of purely growing surface instability for a plasma column is not higher than the threshold of decay (periodic) parametric instability for the same column.

1. Introduction

In the past few years great interest has been aroused in the absorption of intense electromagnetic waves by plasmas through the mechanism of parametric instabilities (Silin 1968, DuBois 1973). Since it is not difficult to achieve the conditions for the excitation of such instabilities an understanding of this process may be of great importance for many plasma experiments. Recently, a number of experimental observations (Franklin *et al* 1971, Chu *et al* 1973) have shown that in real plasma configurations the parametric instabilities arise at frequencies lower than the theoretically predicted ones for infinite plasmas (Nishikawa 1968, Andreev *et al* 1969). On the other hand, the first theoretical work devoted to parametric processes in bounded plasma configurations (Goldman 1969, Aliev and Ferlengi 1969, Aliev *et al* 1972a) shows that the frequency of the pump wave can be near the frequency of the surface waves (for instance such as Trivelpiece–Gould (1959) modes) which can propagate on the system. It is also completely possible for some of the parametrically excited waves to be surface waves as well (Moisan 1971, Chu *et al* 1973). It is worth noting that the threshold for excitation of a purely growing (PG) instability in a plasma layer may prove to be lower than the threshold for the PG instability in infinite plasmas and may be compared with the threshold for decay instability (Aliew *et al* 1972b). In view of the potential possibility for experimental detection (Tamor 1973, Valeo and Oberman 1973) and circumstantial investigation of the PG instability by direct comparison of the experimental data with the results of the theory, it is the purpose of the present paper to derive the threshold conditions and the growth rate of PG instability in a real plasma geometry, namely, a plasma cylinder.

2. Derivation of the basic equations

We shall study the interaction between an external driving electromagnetic field (with a frequency ω_0 and a wavenumber k_0) and a cylinder (with a radius R) of homogeneous isotropic warm plasma bounded by a vacuum. We assume that the electromagnetic wave propagates normally to the cylinder axis as the electric vector $\boldsymbol{E} = \boldsymbol{E}_0 \sin(\omega_0 t - \boldsymbol{k}_0 \cdot \boldsymbol{r})$ is directed along the plasma column axis. If the wavevector $\boldsymbol{k}$ of the parametrically excited waves satisfies the inequality $|\boldsymbol{k}| \gg |\boldsymbol{k}_0|$ (we shall assume that this inequality is satisfied), for the pump wave one may use the dipole approximation, $\boldsymbol{E} = \boldsymbol{E}_0 \sin \omega_0 t$. Moreover, we consider that the parametrically excited modes propagate along the direction of the driving field, ie they are axially-symmetric waves (it is well known that in infinite isotropic plasmas the dominant coupling and the lowest threshold occur when $\boldsymbol{k} \| \boldsymbol{E}_0$ (Markeev 1971, Perkins and Flick 1971).

The dynamics of plasma particles and the field in the plasma are described by the linearized Vlasov and Poisson equations. In the case of bounded plasmas the Vlasov equation can be witten in the form

$$\frac{\partial}{\partial t} f_{1\alpha}(\boldsymbol{r}, \boldsymbol{v}, t) + \boldsymbol{v} \cdot \nabla f_{1\alpha}(\boldsymbol{r}, \boldsymbol{v}, t) + \left(\frac{e_\alpha}{m_\alpha} \boldsymbol{E}_0 \sin \omega_0 t + \frac{\boldsymbol{F}}{m_\alpha}\right) \cdot \frac{\partial}{\partial \boldsymbol{v}} f_{1\alpha}(\boldsymbol{r}, \boldsymbol{v}, t) - \frac{e_\alpha}{m_\alpha} \nabla \varphi(\boldsymbol{r}, t) \cdot \frac{\partial}{\partial \boldsymbol{v}} f_{0\alpha}[\boldsymbol{v}_\perp^2, (\boldsymbol{v}_z - \boldsymbol{v}_{E_\alpha})^2] = 0 \tag{1}$$

where e_α and m_α ($m_e = m$, $m_i = M$) are the charges and the masses of the particles of the species α (α = e, i); $f_{0\alpha}(\boldsymbol{v})$ and $f_{1\alpha}(\boldsymbol{r}, \boldsymbol{v}, t)$ are their equilibrium and perturbed distribution functions; $\varphi(\boldsymbol{r}, t)$ is the potential of the perturbed electric field in the plasma; $\boldsymbol{v}_\perp$ and $\boldsymbol{v}_z$ are the plasma particles' velocity components, perpendicular and parallel to the cylinder axis; $\boldsymbol{v}_{E_\alpha} = -(e_\alpha \boldsymbol{E}_0 / m_\alpha \omega_0) \cos \omega_0 t$ is the oscillating velocity of particles in a field of pump wave $\boldsymbol{E}_0 = (0, 0, E_0)$ and $\boldsymbol{F} = (F, 0, 0)$ is the force which reflects the particles when they strike the plasma–vacuum boundary. The specular reflection of the particles imposes the following condition for the functions $f_{1\alpha}$:

$$f_{1\alpha}(r = R, z, -v_r, \boldsymbol{v}_{\|}, t) = f_{1\alpha}(r = R, z, v_r, \boldsymbol{v}_{\|}, t) \tag{2}$$

(where $|\boldsymbol{v}_{\|}| = (v_z^2 + v_\theta^2)^{1/2}$, which means that $F(-v_r) = -F(v_r)$.

It is relevant that the plasma equations should be written in frames of reference tied to the oscillating plasma particles. In such frames the Vlasov equation has the same form as in a plasma without any external driving field (Dawson and Oberman 1962)

$$\frac{\partial}{\partial t} f_{1\alpha}(\boldsymbol{r}', \boldsymbol{v}', t) + \boldsymbol{v}' \cdot \frac{\partial}{\partial \boldsymbol{r}'} f_{1\alpha}(\boldsymbol{r}', \boldsymbol{v}', t) + \frac{\boldsymbol{F}'}{m_\alpha} \cdot \frac{\partial}{\partial \boldsymbol{v}'} f_{1\alpha}(\boldsymbol{r}', \boldsymbol{v}', t) - \frac{e_\alpha}{m_\alpha} \frac{\partial}{\partial \boldsymbol{r}'} \tilde{\varphi}(\boldsymbol{r}', t) \cdot \frac{\partial}{\partial \boldsymbol{v}'} f_{0\alpha}(\boldsymbol{v}'^2) = 0 \tag{3}$$

where $\boldsymbol{r}'$ is the position vector of the particles in the oscillating frame and

$$\tilde{\varphi}(\boldsymbol{r}', t) = \tilde{\varphi}[\boldsymbol{r} - \boldsymbol{r}_{E_\alpha}(t), t] \tag{4}$$

with $\boldsymbol{r}_{E_\alpha} = -(e_\alpha E_0 / m_\alpha \omega_0^2) \sin \omega_0 t$, is the potential in the same frame. The condition for specular reflection of particles (2) remains invariant with respect to this frame transformation and $\boldsymbol{F}' = \boldsymbol{F}$.

If the dependence of all the quantities on coordinate z' and time t is of the form $\exp[i(k_z'z'-\omega t)]$, kinetic equation (3) written for a cylindrical plasma (after integrating over v_θ) becomes

$$\frac{\partial}{\partial r}f_{1\alpha}(r,v_r)-\mathrm{i}\kappa f_{1\alpha}(r,v_r)+\frac{F}{m_\alpha v_r}\frac{\partial}{\partial v_r}f_{1\alpha}(r,v_r)-e_\alpha f_0'\left(\frac{\partial\tilde{\varphi}}{\partial r}+\mathrm{i}k_z'\tilde{\varphi}\frac{v_z'}{v_r}\right)=0 \tag{5}$$

where $f_0' \equiv \partial f_{0\alpha}/\partial\mathscr{E}_\alpha$ is the derivative of the equilibrium distribution function with respect to energy, and $\kappa \equiv (\omega-k_z'v_z')/v_r$. Boundary condition (2) then simplifies to

$$f_{1\alpha}(R,-v_r)=f_{1\alpha}(R,v_r). \tag{6}$$

It is more convenient to use as solutions of (5) the functions $\psi_\alpha^\pm(r,v_r)$ instead of functions $f_{1\alpha}(r,v_r)$ where

$$\psi_\alpha^\pm(r,v_r)=f_{1\alpha}(r,v_r)\pm f_{1\alpha}(r,v_r).$$

In such a case the boundary condition (6) is homogeneous: $\psi_\alpha^-(R)=0$. Writing the kinetic equation for negative values of v_r and performing addition and subtraction with (5) we obtain two equations for $\psi_\alpha^\pm$

$$\begin{aligned}&\frac{\partial}{\partial r}\psi_\alpha^+-\mathrm{i}\kappa\psi_\alpha^--2e_\alpha f_0'\frac{\partial\tilde{\varphi}}{\partial r}=0\\&\frac{\partial}{\partial r}\psi_\alpha^--\mathrm{i}\kappa\psi_\alpha^+-2e_\alpha f_0'\mathrm{i}k_z'\tilde{\varphi}\frac{v_z'}{v_r}=0.\end{aligned} \tag{7}$$

The radial and axial components of the induced current density $\tilde{j}_\alpha$ are defined through the functions $\psi_\alpha^\pm$ by the expressions

$$\begin{aligned}j_{\alpha r}&=e_\alpha\int_0^\infty \mathrm{d}v_r\int_{-\infty}^\infty \mathrm{d}v_z'\,v_r\psi_\alpha^-(v_r)\\ \tilde{j}_{\alpha z'}&=e_\alpha\int_0^\infty \mathrm{d}v_r\int_{-\infty}^\infty \mathrm{d}v_z'\,v_z'\psi_\alpha^+(v_r).\end{aligned} \tag{8}$$

We shall seek the solution of (7) by making series expansions for functions ψ_α^+ and $\tilde{\varphi}$ in Dini's series

$$\psi_\alpha^+(r)=\frac{2}{R^2}\sum_{\nu=1}^\infty A_\nu^+(\xi_\nu)\frac{J_0(r\xi_\nu)}{|J_0(R\xi_\nu)|^2} \tag{9a}$$

$$\tilde{\varphi}(r)=\frac{2}{R^2}\sum_{\nu=1}^\infty \tilde{\phi}(\xi_\nu)\frac{J_0(r\xi_\nu)}{|J_0(R\xi_\nu)|^2} \tag{9b}$$

and for ψ_α^- in Fourier–Bessel series (Petiau 1955)

$$\psi_\alpha^-(r)=\frac{2}{R^2}\sum_{\nu=1}^\infty A_\alpha^-(\xi_\nu)\frac{J_1(r\xi_\nu)}{|J_1'(R\xi_\nu)|^2} \tag{9c}$$

where $\xi_\nu \geqslant 0$ are roots of the equation $J_1(R\xi)=0$; J_n are the Bessel functions of the first kind and nth order; the prime means differentiation with respect to the argument of the function.

After multiplying equations (7) by $rJ_1(r\xi_\mu)$ and $rJ_0(r\xi_\mu)$, respectively, and integrating over r from 0 to R, one obtains a set of equations for the coefficients $A_\alpha^\pm(\xi_\nu)$ which may be expressed in terms of $\tilde{\phi}(\xi_\nu)$. If for the current densities $\tilde{j}_{\alpha r(z')}$ we make series expansions similar to those for $\psi_\alpha^\mp$, we find the following relations between the coefficients $\tilde{j}_{\alpha r(z')}(\xi_\mu)$ and $\tilde{\phi}(\xi_\mu)$:

$$\begin{aligned} \tilde{j}_{\alpha r}(\xi_\mu) &= 2\mathrm{i}\omega e_\alpha^2 \xi_\mu \left(\int_{-\infty}^{\infty} \mathrm{d}v_z' \int_0^\infty \frac{\mathrm{d}v_r\, f_0'}{\xi_\mu^2 - \kappa^2} \right) \tilde{\phi}(\xi_\mu) \\ \tilde{j}_{\alpha z'}(\xi_\mu) &= 2e_\alpha^2 \left(\int_{-\infty}^{\infty} \mathrm{d}v_z'\, v_z' \int_0^\infty \frac{\mathrm{d}v_r\, f_0'(\xi_\mu^2 + \kappa k_z' v_z'/v_r)}{\xi_\mu^2 - \kappa^2} \right) \tilde{\phi}(\xi_\mu) \end{aligned} \tag{10}$$

provided that the contribution of the integral $\int_0^R \mathrm{d}r\, \psi^-(r) J_0(r\xi_\mu)$ appearing in the calculations, is ignored. (A justification for this is given in the appendix.)

The continuity equation for charge

$$\frac{\partial \tilde{\rho}_\alpha}{\partial t} + \operatorname{div} \tilde{\boldsymbol{j}}_\alpha = 0$$

written in the space of the variables ξ_μ gives us a certain relation between the charge density perturbation $\tilde{\rho}_\alpha(\xi_\mu)$ and the potential $\tilde{\phi}(\xi_\mu)$. Taking this relation into account, on the basis of the equality

$$4\pi\tilde{\rho}_\alpha(\xi_\mu) = -\delta\mathscr{E}_\alpha(\omega, \xi_\mu, k_z')(\xi_\mu^2 + k_z'^2)\tilde{\phi}(\xi_\mu) \tag{11}$$

one can define the partial longitudinal dielectric permittivities (electric susceptibilities) of the particles of the species α

$$\begin{aligned} \delta\mathscr{E}_\alpha(\omega, \xi_\mu, k_z') = -\frac{8\pi e_\alpha^2}{\xi_\mu^2 + k_z'^2} \Bigg(\frac{k_z'}{\omega} \int_{-\infty}^{\infty} \mathrm{d}v_z'\, v_z' \int_0^\infty \frac{\mathrm{d}v_r\, f_0'(\xi_\mu^2 + \kappa k_z' v_z'/v_r)}{\xi_\mu^2 - \kappa^2} \\ + \xi_\mu^2 \int_{-\infty}^{\infty} \mathrm{d}v_z' \int_0^\infty \frac{\mathrm{d}v_r\, f_0'}{\xi_\mu^2 - \kappa^2} \Bigg). \end{aligned} \tag{12}$$

The potential of the self-consistent electric field which arises in the laboratory frame as a result of the interaction of the driving field with the plasma is determined from the Poisson equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \varphi}{\partial r} \right) + \frac{\partial^2 \varphi}{\partial z^2} = -4\pi \sum_\alpha e_\alpha \int \mathrm{d}\boldsymbol{v}\, f_{1\alpha} \tag{13}$$

whose solution will be sought in the form

$$\varphi(\boldsymbol{r}, t) = \sum_{n=-\infty}^{\infty} \varphi^{(n)}(r) \exp[-\mathrm{i}(\omega + n\omega_0)t + \mathrm{i}k_z z] \tag{14}$$

where

$$\varphi^{(n)}(r) = \begin{cases} A_1^{(n)} K_0(k_z r) & \text{at } r > R \\ A_2^{(n)} I_0(k_z r) + \dfrac{2}{R^2} \displaystyle\sum_{\nu=1}^{\infty} F^{(n)}(\xi_\nu) \frac{J_0(r\xi_\nu)}{|J_0(R\xi_\nu)|^2} & \\ = \dfrac{2}{R^2} \displaystyle\sum_{\nu=1}^{\infty} \phi^{(n)}(\xi_\nu) \frac{J_0(r\xi_\nu)}{|J_0(R\xi_\nu)|^2} & \text{at } r < R \end{cases} \tag{15}$$

I_n and K_n are the modified Bessel functions of the first and second kind and nth order. Following Cheng and Harris (1969) we choose the general solution of (13) within the plasma as a sum of a solution of Laplace's equation and a solution of the Poisson equation assuming that the parametrically excited waves are surface modes. From (15) one may derive a recurrence formula between $F^{(n)}(\xi_\nu)$ and $\phi^{(n)}(\xi_\nu)$

$$\phi^{(n)}(\xi_\nu) = J_0(R\xi_\nu)I_1(k_zR)\frac{k_zR}{\xi_\nu^2+k_z^2}A_2^{(n)} + F^{(n)}(\xi_\nu). \tag{16}$$

After performing a Fourier transform in time t and space coordinate z, equation (13) according to (14) and (15) takes the form

$$(\xi_\mu^2+k_z^2)F^{(n)}(\xi_\mu, k_z, \omega+n\omega_0) = 4\pi\sum_\alpha \rho_\alpha^{(n)}(\xi_\mu, k_z, \omega+n\omega_0). \tag{17}$$

The beat Fourier components $\rho_\alpha^{(n)}(\xi_\mu, k_z, \omega)$ of the charge density $\rho_\alpha(\boldsymbol{r}, t) \equiv e_\alpha\int \mathrm{d}\boldsymbol{v}\, f_{1\alpha}$ can be expressed through the Fourier components of the charge density in the oscillating frame. Bearing in mind that the oscillating and laboratory frames are equivalent with respect to wave propagation in a plasma (due to the directed motion of the particles of some α species) with and without a driving field, respectively, it is possible to write

$$\rho_\alpha(\boldsymbol{r}'+\boldsymbol{r}_{E_\alpha}(t), t) = \tilde{\rho}_\alpha(\boldsymbol{r}', t).$$

A Fourier transform on t and z and Fourier–Bessel transform on r (taking into account that $\boldsymbol{r}_{E_\alpha}$ has only one component along the z axis and $k_z' = k_z$) reduce the above equality to

$$\begin{aligned}\tilde{\rho}_\alpha(\xi_\mu, k_z, \omega) &= \frac{\mathrm{i}}{2\pi}\sum_{p=-\infty}^{\infty} J_p(a_\alpha)\int_{-\infty}^{\infty}\mathrm{d}\omega'\,\frac{\rho_\alpha(\xi_\mu, k_z, \omega')}{\omega-\omega'-p\omega_0} \equiv \Delta_{E_\alpha}\cdot\rho_\alpha(\xi_\mu, k_z, \omega')\\ \rho_\alpha(\xi_\mu, k_z, \omega) &= \frac{\mathrm{i}}{2\pi}\sum_{p=-\infty}^{\infty} J_p(a_\alpha)\int_{-\infty}^{\infty}\mathrm{d}\omega'\,\frac{\tilde{\rho}_\alpha(\xi_\mu, k_z, \omega')}{\omega-\omega'+p\omega_0} \equiv \tilde{\Delta}_{E_\alpha}\cdot\tilde{\rho}_\alpha(\xi_\mu, k_z, \omega')\end{aligned} \tag{18}$$

where $a_\alpha = k_z e_\alpha E_0/m_\alpha\omega_0^2$ (Lee *et al* 1972). An analogous relation exists also for the potentials $\phi(\xi_\mu)$ and $\tilde{\phi}(\xi_\mu)$. Since $\tilde{\rho}_\alpha(\xi_\mu)$ are expressed in terms of $\delta\mathscr{E}_\alpha$ and $\tilde{\phi}(\xi_\mu)$ (equation (11)), the right-hand side of the Poisson equation becomes

$$4\pi\rho_\alpha(\xi_\mu, k_z, \omega) = 4\pi\sum_\alpha \tilde{\Delta}_{E_\alpha}\cdot\tilde{\rho}_\alpha(\xi_\mu, k_z, \omega') = -(\xi_\mu^2+k_z^2)\sum_\alpha \tilde{\Delta}_{E_\alpha}\cdot[\delta\mathscr{E}_\alpha(\xi_\mu, k_z, \omega')\tilde{\phi}(\xi_\mu)].$$

When the operator Δ_{E_α} is applied to $\phi(\xi_\mu)$, equation (17) reads as follows

$$F(\xi_\mu, k_z, \omega) = \sum_\alpha \tilde{\Delta}_{E_\alpha}\cdot[\delta\mathscr{E}_\alpha(\xi_\mu, k_z, \omega')\Delta_{E_\alpha}\cdot\phi(\xi_\mu, k_z, \omega'')]. \tag{19}$$

According to the kind of operators Δ_{E_α} and $\tilde{\Delta}_{E_\alpha}$ equation (19) after substituting $\omega+n\omega_0$ for ω transforms to

$$\begin{aligned}&F^{(n)}(\xi_\mu, k_z, \omega+n\omega_0)\\ &\quad= -\sum_\alpha\sum_{p,s=-\infty}^{\infty}\phi^{(p)}(\xi_\mu, k_z, \omega+p\omega_0)\,\delta\mathscr{E}_\alpha^{(s)}(\xi_\mu, k_z, \omega+s\omega_0)J_{s-n}(a_\alpha)J_{s-p}(a_\alpha).\end{aligned} \tag{20}$$

When the boundary effects vanish (infinite plasmas) and $\phi^{(p)}$ tends to $F^{(p)}$, from (20) one can derive the dispersion relation of the waves in a plasma immersed in an external high-frequency electric field (Aliev and Silin 1965, Sanmartin 1970, Kaw and Dawson 1971).

The conditions for continuity of the potential φ and its normal derivative $\partial\varphi/\partial r$ at the plasma–vacuum boundary are

$$\varphi^{\mathrm{v}}(R) = \varphi^{\mathrm{p}}(R) \qquad \text{and} \qquad \left.\frac{\partial\varphi^{\mathrm{v}}}{\partial r}\right|_{r=R} = \left.\frac{\partial\varphi^{\mathrm{p}}}{\partial r}\right|_{r=R}. \tag{21}$$

According to (15) conditions (21) reduce to the set of equations

$$A_2^{(n)} I_0'(k_z R)\left(1 - \frac{I_0(k_z R) K_0'(k_z R)}{I_0'(k_z R) K_0(k_z R)}\right) = \frac{K_0'(k_z R)}{K_0(k_z R)} \frac{2}{R^2} \sum_{\nu=1}^{\infty} F^{(n)}(\xi_\nu) \frac{1}{J_0(R\xi_\nu)}. \tag{22}$$

The determination of the beat coefficients $F^{(n)}(\xi_\nu)$ by means of recurrence formula (16) and equation (20) yields an infinite set of equations for $F^{(n)} (n = 1, 2, \ldots)$

$$\frac{2}{R^2} \frac{F^{(n)}}{J_0(R\xi_\nu)} + \sum_\alpha \sum_{p,s=-\infty}^{\infty} \frac{2}{R^2} \frac{F^{(p)}}{J_0(R\xi_\nu)} \delta\mathscr{E}_\alpha^{(s)} J_{s-n}(a_\alpha) J_{s-p}(a_\alpha)$$

$$= -\sum_\alpha \sum_{p,s=-\infty}^{\infty} \frac{2k_z I_1(k_z R)}{R(\xi_\nu^2 + k_z^2)} A_2^{(p)} \delta\mathscr{E}_\alpha^{(s)} J_{s-n}(a_\alpha) J_{s-p}(a_\alpha). \tag{23}$$

From the solvability condition of (22) and (23) one obtains the dispersion relation of the waves in a warm plasma column bounded by a vacuum and driven by an external high-frequency electric field.

3. Purely growing surface parametric instability under the interaction of a strong high-frequency electric field with a plasma column

If the driving field is strong, $v_{E_e} \gg v_{T_e}$ ($v_{E_e} = eE_0/m\omega_0$ is the amplitude of the oscillating electrons' velocity and $v_{T_e} = (T_e/m)^{1/2}$ is their thermal velocity), and the frequency ω_0 is higher than the ion plasma frequency, $\omega_0 \gg \omega_{pi} = (4\pi e^2 n_0/M)^{1/2}$, the ions' oscillating velocity can be considered equal to zero and the dissipative losses due to the particles' thermal motion (Landau damping) may be ignored. In such a case $\omega \gg kv_{T_e}$ and the partial dielectric permittivities of the electrons and ions are, respectively,

$$\delta\mathscr{E}_e^{(s)} = -\frac{\omega_{pe}^2}{(\omega + s\omega_0)^2} \quad \text{and} \quad \delta\mathscr{E}_i^{(s)} = -\frac{\omega_{pi}^2}{(\omega + s\omega_0)^2} \simeq 0 \qquad (\text{for } s \neq 0).$$

Here $\omega_{pe} = (4\pi e^2 n_0/m)^{1/2}$ is the electron plasma frequency. After summing over ν in (23) and substituting the left-hand side of (22) for

$$\frac{2}{R^2} \sum_{\nu=1}^{\infty} F^{(n)}(\xi_\nu) \frac{1}{J_0(R\xi_\nu)}$$

in (23), we obtain the following set of equations:

$$\left(1 - \frac{I_0(k_z R) K_0'(k_z R)}{I_0'(k_z R) K_0(k_z R)}\right) \left[A_2^{(n)} + \sum_\alpha \sum_{p,s=-\infty}^{\infty} A_2^{(p)} \delta\mathscr{E}_\alpha^{(s)} J_{s-n}(a_\alpha) J_{s-p}(a_\alpha) \left(1 - \frac{I_0(k_z R) K_0'(k_z R)}{I_0'(k_z R) K_0(k_z R)}\right)^{-1}\right]$$
$$= 0.$$

The solution of the infinite determinant corresponding to this set yields the spectra derived by Aliev and Ferlengi (1969). The various cases for PG instability under the interaction of a plasma column or layer with a strong high-frequency electric field are analysed in detail in their paper. We note that the results of Aliev and Ferlengi are valid whenever the growth rates of the discovered instabilities are greater than $k_z v_{T_e}$. It is obvious that the threshold fields (for a PG surface instability or for a periodic low-frequency instability, both accompanied with high-frequency modes) and the growth rate near the threshold can be derived in the approximation of a weak field.

4. Purely growing surface parametric instability under the interaction of a weak high-frequency electric field with a warm plasma column

If the driving field is weak, $v_{E_e} \ll v_{T_e}$, its frequency ω_0 is near the frequency ω_s of the high-frequency surface waves and the wavelength of the instabilities $2\pi/k_z$ is much greater than the Debye radius of the electrons, ie if $(\omega_{pe}/\omega_0)k_z r_{De} < 1$ where $r_{D\alpha} = (T_\alpha/4\pi e^2 n_0)^{1/2}$, the frequency ω of the low-frequency component of the resulting PG instability is much lower than ω_0 and $k_z v_{T_\alpha}$. In this case only the Bessel functions $J_n(a_\alpha)$ with indices $n = 0, \pm 1$ and the partial dielectric permittivities $\delta\mathscr{E}_e^{(0,\pm 1)}$ give considerable contributions to the terms of equation (23). Since $a_i \simeq 0$, in the leading terms we shall retain $\delta\mathscr{E}_i^{(0)}$ and $J_0(a_i) = 1$ only. Such a truncation of the set of equations is possible only if the waves corresponding to the beat Fourier components $F^{(0,\pm 1)}$ are resonantly excited and their dispersion relation is nearly satisfied. On the other hand these waves can be simultaneously resonantly excited by the parametric interaction if the frequency and wavevector matching conditions (the laws of conservation of quasi-energy and quasi-momentum)

$$\omega_{\boldsymbol{k}} = \omega_{\boldsymbol{k}'} + \omega_{\boldsymbol{k}-\boldsymbol{k}'}$$
$$\boldsymbol{k} = \boldsymbol{k}' + (\boldsymbol{k}-\boldsymbol{k}')$$

are satisfied. In the dipole approximation for the pump field, the wave with a frequency $(\omega_0 - \omega)$ which corresponds to $F^{(-1)}(k_z)$ has a wavenumber $-k_z$, while the waves with frequencies ω and $\omega_0 + \omega$, corresponding to $F^{(0)}(k_z)$ and $F^{(1)}(k_z)$, have a wavenumber k_z.

The calculation of the coefficients $F^{(0,\pm 1)}$ from (23) and their substitution in (22) results in a homogeneous set of equations for the coefficients $A_2^{(0,\pm 1)}$. The condition for solvability of this set yields the dispersion relation of parametric instabilities in a warm plasma column bounded by a vacuum

$$D_0(\omega, k_z) + \frac{a_e^2}{4}\left(\frac{1}{D_1\mathscr{E}^{(1)}} + \frac{1}{D_{-1}\mathscr{E}^{(-1)}}\right) S(\omega, k_z) = 0 \qquad (24)$$

where

$$D_n(\omega, k_z) = 1 + \frac{K_1(k_z R)}{K_0(k_z R)} \frac{2k_z}{R} \sum_{\nu=1}^{\infty} \frac{1}{(\xi_\nu^2 + k_z^2)\mathscr{E}^{(n)}} \qquad n = 0, \pm 1$$

$$\mathscr{E}^{(n)}(\omega, k_z) = 1 + \delta\mathscr{E}_e^{(n)} + \delta\mathscr{E}_i^{(n)}$$

$$S(\omega, k_z) = \frac{I_1(k_z R)}{I_0(k_z R)} \frac{2k_z}{R} \sum_{\nu=1}^{\infty} \frac{\delta\mathscr{E}_e^{(0)}\delta\mathscr{E}_i^{(0)}}{(\xi_\nu^2 + k_z^2)\mathscr{E}^{(0)}}.$$

Taking the definition of D_n's into account the sum in the bracket in (24) reads

$$\frac{1}{D_1\mathscr{E}^{(1)}} + \frac{1}{D_{-1}\mathscr{E}^{(-1)}} = \left(1 + \frac{I_0(k_z R)K_1(k_z R)}{I_1(k_z R)K_0(k_z R)}\right)^{-1} \frac{\omega_0 \Delta}{\Delta^2 - (\omega + \mathrm{i}\tilde{\gamma})^2} \qquad (25)$$

where $\Delta = \omega_0 - \omega_s$ is the mismatch of the driving field, $\tilde{\gamma}$ is the damping rate of the high-frequency surface waves with a frequency ω_s.

In the range of low frequencies when $\omega \ll k_z v_{T_\alpha}$, the partial dielectric permittivities $\delta\mathscr{E}_\alpha(\omega, k)$ in a collisionless plasma with Maxwellian equilibrium distribution functions according to (12) take the form

$$\delta\mathscr{E}_\alpha(\omega, k) = \frac{1}{k^2 r_{D\alpha}^2}\left[1 + \mathrm{i}\left(\frac{\pi}{2}\frac{m_\alpha}{T_\alpha}\right)^{1/2} \frac{\omega}{k}\right] \qquad (26)$$

where $k \equiv (\xi_\nu^2+k_z^2)^{1/2}$. As expression (26) coincides with the well known formula for $\delta\mathscr{E}_\alpha$ of infinite fully ionized plasmas (Ginzburg and Rukhadze 1970), we assume that the contribution of the collisional dissipation to $\delta\mathscr{E}_\alpha$ for a bounded (cylindrical) plasma is the same as for an infinite one:

$$\delta\mathscr{E}_\alpha(\omega, k) = \frac{1}{k^2 r_{D\alpha}^2}\left\{1+\mathrm{i}\frac{\omega}{kv_{T_\alpha}}\left[\left(\frac{\pi}{2}\right)^{1/2}+\frac{\nu_{\mathrm{eff}}}{kv_{T_\alpha}}\right]\right\}. \tag{27}$$

When $T_i/T_e > (m/M)^{1/2}$ the ion collisional dissipation can be neglected and the dissipation due to the electrons can be specified by the magnitude of ν_{eff}. In a fully ionized plasma $\nu_{\mathrm{eff}} = 1{\cdot}44\,\nu_{ei}$ (Bogdankevich *et al* 1967) where

$$\nu_{ei} = \frac{4}{3}\left(\frac{2\pi}{m}\right)^{1/2}\frac{e^4 L n_0}{T_e^{3/2}}$$

and L is the Coulomb logarithm (Ginzburg and Rukhadze 1970).

Having derived the formula for $\delta\mathscr{E}_\alpha$ an expression for D_0 may then be found

$$D_0(\omega, k_z) = 1+\frac{K_1(k_zR)}{K_0(k_zR)}\frac{2k_z}{R}\sum_{\nu=1}^{\infty}\frac{1}{\xi_\nu^2+k_z^2+r_D^{-2}}-\mathrm{i}\omega\chi(k_z, v_{T_\alpha}, \nu_{\mathrm{eff}}) \tag{28}$$

where the quantity χ has the values:

$$\chi_{\mathrm{collisionless}} = \frac{K_1(k_zR)}{K_0(k_zR)}\frac{2k_z}{R}\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{1}{v_{T_e}r_{De}^2}+\frac{1}{v_{T_i}r_{Di}^2}\right)\sum_{\nu=1}^{\infty}\frac{1}{(\xi_\nu^2+k_z^2)^{1/2}(\xi_\nu^2+k_z^2+r_D^{-2})^2} \tag{29a}$$

for collisionless plasmas, and

$$\chi_{\mathrm{collisional}} = \frac{K_1(k_zR)}{K_0(k_zR)}\left[\left(\frac{\pi}{2}\right)^{1/2}\frac{1}{v_{T_i}r_{Di}^2}\frac{2k_z}{R}\sum_{\nu=1}^{\infty}\frac{1}{(\xi_\nu^2+k_z^2)^{1/2}(\xi_\nu^2+k_z^2+r_D^{-2})^2}\right.$$
$$\left.+\frac{\nu_{\mathrm{eff}}}{v_{T_e}^2 r_{De}^2}\frac{2k_z}{R}\sum_{\nu=1}^{\infty}\frac{1}{(\xi_\nu^2+k_z^2)(\xi_\nu^2+k_z^2+r_D^{-2})^2}\right] \tag{29b}$$

for plasmas with collisions. In these expressions

$$r_D = \left(\frac{1}{r_{De}^2}+\frac{1}{r_{Di}^2}\right)^{-1/2}$$

is the plasma Debye radius.

It is well known that when there are no external pump fields, nearly self-consistent potential waves with frequencies $\omega < k_z v_{T_\alpha}$ cannot propagate in a plasma. However, in the presence of a pump field according to (24) Im D_0 can alter its sign for a fixed value of k_z which is equivalent to the appearance of PG instability (with a frequency Re $\omega = 0$ and a growth rate γ = Im $\omega > 0$) provided that the pump amplitude is greater than a threshold value E_0^{thr}:

$$\eta \equiv \frac{(E_0^{\mathrm{thr}})^2}{4\pi n_0 T_e} = 8\left(1+\frac{T_i}{T_e}\right)\left(\frac{\omega_0}{\omega_{pe}}\right)^4\left(\frac{I_0(k_zR)}{K_0(k_zR)}\right)\left(\frac{I_1(k_zR)K_0(k_zR)+I_0(k_zR)K_1(k_zR)}{[I_0^2(k_zR)-I_1^2(k_zR)]k_zR}\right)$$
$$\times\left(\frac{\Delta^2+\tilde{\gamma}^2}{\omega_0\Delta}\right). \tag{30}$$

The minimum threshold occurs when $|\Delta| = \tilde{\gamma}$ and if $k_zR \to \infty$ (semi-bounded plasma) the value of $\eta_{\min}$ coincides with the threshold derived by Aliev *et al* (1972a).

As ω_s and $\tilde{\gamma}$ are functions of k_z the condition $|\Delta| = \tilde{\gamma}$ specifies the threshold wavelength, $2\pi/k_z^{\text{thr}}$, of the PG instability with its onset.

The growth rate of the PG instability may be derived from (24) by substituting $-\mathrm{i}\omega = \gamma$ and we obtain

$$\gamma = \frac{\Delta^2+\tilde{\gamma}^2}{2\tilde{\gamma}}\left(\frac{\eta-\eta_{\min}}{\eta_{\min}}\right)\left(1+\frac{\chi(\Delta^2+\tilde{\gamma}^2)}{2\tilde{\gamma}}\right)^{-1}. \tag{31}$$

From an experimental point of view it is desirable to start an analysis of formulae (30) and (31) for reasonable sizes of the plasma column. Let us calculate the threshold field for a thin cylinder, $k_z R < 1$. (The threshold for a thick cylinder, $k_z R > 1$, is practically the same as for a semi-bounded plasma.) According to equation (24) with $\delta\mathscr{E}_i^{(0)} = 0$ and $a_e = 0$ the spectrum of the surface waves in the frequency range $\omega_{pe} > \omega_s > \omega_{pi}$, $k_z v_{T_e}$, ν_{eff} is given by

$$\omega_s = \omega_{pe} k_z R[\ln(k_z R)^{-1/2}]^{1/2} \tag{32}$$

and their damping rate is

$$\tilde{\gamma} = \frac{k_z R}{4\sqrt{\pi}} k_z v_{T_e} \ln\frac{\omega_{pe}}{\omega_s} + \frac{\nu_{eff}}{2}.\dagger \tag{33}$$

The appearance of the small term $\frac{1}{2}\nu_{eff}$ (where for fully ionized plasmas $\nu_{eff} = \nu_{ei}$) is equivalent to taking into account the collisions in the kinetic equation through a term $(\delta f_{1\alpha}/\delta t)_{coll} \to -\nu_{eff} f_{1\alpha}$. Furthermore the collisional damping rate in (33) is derived on the assumption that the plasma boundedness cannot influence the collisional mechanism among the particles.

On the basis of the values derived for ω_s and $\tilde{\gamma}$, from formula (30) one finds the final expression for the threshold field of the PG surface instability in a thin plasma column

$$\eta_{\min} = 2\sqrt{2}\left(1+\frac{T_i}{T_e}\right)k_z R\left(\ln\frac{1}{k_z R}\right)^{1/2}\left(\frac{k_z^2 R r_{De}}{2\sqrt{\pi}}\ln\frac{\sqrt{2}}{k_z R[\ln(k_z R)^{-1}]^{1/2}}+\frac{\nu_{eff}}{\omega_{pe}}\right). \tag{34}$$

Obviously, for some values of ν_{eff} the collisional damping determines the threshold for the instability onset. Moreover in a collisional plasma $\eta_{\min}$ depends linearly on the plasma radius while in collisionless plasmas this dependence is quadratic.

The growth rate of the PG surface instability in the super-threshold regime can be calculated in the case of collisional plasmas only. For the reasonable values of the mismatch (ie $|\Delta| = \tilde{\gamma}$) the second term in the bracket of (31), which includes the kinetic effects as well, according to (29*b*) and (33) becomes

$$\chi_{\text{collisional}}\tilde{\gamma} = \frac{K_1(k_z R)}{K_0(k_z R)}\left(\frac{\nu_{eff}}{\omega_{pe}}\right)^2\left(1+\frac{T_e}{T_i}\right)^{-2}\left(\frac{1}{2\sqrt{\pi}}\frac{k_z v_{T_e}}{\nu_{eff}}\ln\frac{\omega_{pe}}{\omega_s}+\frac{1}{k_z R}\right).$$

If in the plasma the hydrodynamical effects are essential ($\nu_{eff} > k_z v_{T_e}$), bearing in mind that $\ln \omega_{pe}/\omega_s < 1/k_z R$, one may conclude that the inequality $\chi_{\text{collisional}}\tilde{\gamma} \ll 1$ is always satisfied provided that $(\nu_{eff}/\omega_{pi})^2 < (1+T_e/T_i)^2$. The latter inequality is obtained when $k_z R$ takes its minimum value derived from the condition $\omega_s \gtrsim \omega_{pi}$. Hence the maximum growth rate of the PG instability with the superthreshold values of the field is

† The value of $\tilde{\gamma}$ adduced in Zhelyazkov and Nenovski (1973) is wrong because of incorrect reading of formula (13) in the paper by Kondratenko (1972). (The check-up shows that the true value of $\tilde{\gamma}$ must be taken from (33).)

reached for wavenumbers $k_z = k_z^{thr}$ and has a magnitude

$$\gamma_{max} = \frac{\eta - \eta_{min}}{\eta_{min}} \tilde{\gamma}. \tag{35}$$

In the case of collisionless plasmas when the kinetic effects dominate ($\nu_{eff} < k_z v_{T_e}$) we cannot calculate the exact value of $\chi_{collisionless}$. But estimating this quantity and taking its minimum value we obtain the inequality

$$\chi_{collisionless}\tilde{\gamma} \geqslant \frac{1}{4\sqrt{2}}\left(\frac{M}{m}\right)^{1/2}\left(\frac{\omega_{pe}}{\omega_s}\right)^2 k_z^3 R r_{De} r_{Di}\left(\ln\frac{\omega_{pe}}{\omega_s}\right)\frac{1+(m/M)^{1/2}(T_i/T_e)^{3/2}}{(1+T_i/T_e)^2}.$$

Since the value of this product can be greater than unity, the instability growth rate would be lower than the growth rate for a collisional plasma. Furthermore, the wavenumber for which the growth rate is maximum would be different from k_z^{thr}. However, since for real plasmas we always have $\nu_{eff} \neq 0$, the instability with the maximum growth rate (whose wavenumber satisfies the inequality $k_z^{max} v_{T_e} < \nu_{eff}$) will occur, ie even in the threshold regime the dominant effects are hydrodynamical.

5. Concluding remarks

Dispersion relation (24) may also be examined for the case of decay parametric instability when the pump frequency ω_0 is greater than ω_s and the parametrically excited high- and low-frequency waves are surface modes. If the low-frequency component of the decay instability had a real frequency of the order of the ion-acoustic one, $\omega_{ac} = k_z(T_e/M)^{1/2}$, the threshold η_{min}^D of such an instability would be

$$\eta_{min}^D = \eta_{min}^{PG}\frac{\tilde{\gamma}\gamma_{ac}}{\omega_{ac}^2}.$$

As the damping rate of the surface ion-acoustic wave γ_{ac} is greater than (or equal to) the damping rate of the bulk ion-acoustic waves, according to the value of $\tilde{\gamma}$ taken from (33), one obtains $\tilde{\gamma}\gamma_{ac}/\omega_{ac}^2 \geqslant 1$. Therefore, the threshold η_{min}^D for the onset of surface decay parametric instability is not lower than the threshold η_{min}^{PG} for PG surface parametric instability, while in the case of infinite systems for non-isothermal plasmas, $T_e > T_i$, we have bulk waves and always $\eta_{min}^{PG} > \eta_{min}^D$ (Andreev *et al* 1969) and for isothermal plasmas, $T_e \simeq T_i$, the two thresholds are equal (DuBois and Goldman 1967).

When the plasma is infinite (or semi-bounded) the existence of oscillations (waves in the long wavelength limit $k \to 0$) with certain characteristic frequency $\omega_{char}(k = 0)$ is possible. This allows us to fix the sign of the mismatch $\Delta = \omega_0 - \omega_{char}$ and by suitable adjustment of Δ we may control the nature of instability. For a plasma column the limit $k_z \to 0$, however, is not reasonable and since we cannot in advance know the instability wavenumber, we are not able to find the value of $\omega_s(k_z)$. In other words, we cannot control the sign of the mismatch $\Delta = \omega_0 - \omega_s$. In a bounded plasma system the instability with maximum growth rate will be excited and the nature of this instability may be identified only experimentally.

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Appendix

For convenience we rewrite the function $\psi^-(r)$ (see (9*c*)) in the following way

$$\psi^-(r) = \sum_{\nu=1}^{\infty} A_\nu^- J_1(r\xi_\nu). \tag{A.1}$$

As $\psi^-(0) = \psi^-(R) = 0$ expression (A.1) possesses at least one extremum. We assume that it is unique:

$$\psi_{\rm extr}^- = \psi^-(r_{\rm extr}) \neq 0 \qquad 0 < r_{\rm extr} < R. \tag{A.2}$$

(The case with several extrema remains in principle the same.) The analysis of kinetic equation (7) shows that in ξ's space one obtains an infinite set of algebraic equations for the coefficients $A_\mu^\pm (\mu = 1, 2, \ldots)$. Since each equation contains all the coefficients, this set is practically unsolvable.

Let us now estimate qualitatively the contribution of the term $\int_0^R \mathrm{d}r\, \psi^-(r) J_0(r\xi_\mu)$ to $\mathrm{Re}\, \mathscr{E}^l(\omega, k)$ and $\mathrm{Im}\, \mathscr{E}^l(\omega, k)$, ie to the spectrum and the damping rate of the waves which can propagate ($\mathscr{E}^l(\omega, k) = 0$). According to the assumption for one extremum, (A.2), using Bonné's theorem for average values, we derive that

$$\int_0^R \mathrm{d}r\, \psi^-(r) J_0(r\xi_\mu) = \psi^-(r_{\rm extr}) \int_{r_1}^{r_2} \mathrm{d}r\, J_0(r\xi_\mu) = \psi^-(r_{\rm extr})\Lambda_\mu \tag{A.3}$$

where $0 < r_1 < r_{\rm extr} < r_2 < R$. Then

$$\begin{aligned} A_\mu^- &= \frac{\xi_\mu}{\kappa^2 - \xi_\mu^2}\left((\mathrm{i}\kappa\delta_1 + \delta_2)\phi_\mu - \frac{1}{R^2}\int_0^R \mathrm{d}r\, \psi^-(r) J_0(r\xi_\mu)\right) \\ &= \frac{\xi_\mu}{\kappa^2 - \xi_\mu^2}\left((\mathrm{i}\kappa\delta_1 + \delta_2)\phi_\mu - \frac{1}{R^2}\psi^-(r_{\rm extr})\Lambda_\mu\right) \end{aligned} \tag{A.4}$$

where $\delta_1 = 2e_\alpha f'_0$ and $\delta_2 = 2e_\alpha f'_0 \mathrm{i} k'_z v'_z / v_r$. Multiplying the two sides of (A.4) by $J_1(r\xi_\mu)$ and summing over μ from 1 to ∞ we find that its left-hand side is exactly $\psi^-(r)$. Further we determine $\psi^-(r_{\rm extr})$ (using (A.3)):

$$\begin{aligned} \psi^-(r_{\rm extr}) &= (\mathrm{i}\kappa\delta_1 + \delta_2) \sum_{\mu=1}^{\infty} \frac{\xi_\mu \phi_\mu}{\kappa^2 - \xi_\mu^2} \int_0^R \mathrm{d}r\, J_1(r\xi_\mu) J_0(r\xi_\nu) \\ &\quad \times \left(\Lambda_\nu + \frac{1}{R^2}\sum_{\lambda=1}^{\infty} \frac{\xi_\lambda \Lambda_\lambda}{\kappa^2 - \xi_\lambda^2} \int_0^R \mathrm{d}r\, J_1(r\xi_\lambda) J_0(r\xi_\nu)\right)^{-1}. \end{aligned} \tag{A.5}$$

One may show that by substituting $\psi^-(r_{\rm extr})$ in (A.4) the coefficients A_μ^- can be presented in the form

$$A_\mu^- = \alpha_{\mu\mu}\phi_\mu + \sum_{\substack{\beta=1 \\ \beta \neq \mu}}^{\infty} \alpha_{\mu\beta}\phi_\beta \tag{A.6}$$

where

$$\alpha_{\mu\mu} = (\mathrm{i}\kappa\delta_1+\delta_2)\frac{\xi_\mu}{\kappa^2-\xi_\mu^2}\left[1-\left(\frac{\Lambda_\mu}{R^2}\frac{\xi_\mu}{\kappa^2-\xi_\mu^2}\int_0^R \mathrm{d}r\, J_1(r\xi_\mu)J_0(r\xi_\mu)\right)\right.$$
$$\left.\times\left(\Lambda_\mu+\frac{1}{R^2}\sum_{\lambda=1}^{\infty}\frac{\xi_\lambda\Lambda_\lambda}{\kappa^2-\xi_\lambda^2}\int_0^R \mathrm{d}r\, J_1(r\xi_\lambda)J_0(r\xi_\mu)\right)^{-1}\right]$$

$$\alpha_{\mu\beta} = -(\mathrm{i}\kappa\delta_1+\delta_2)\frac{\xi_\mu\xi_\beta}{(\kappa^2-\xi_\mu^2)(\kappa^2-\xi_\beta^2)}\frac{\Lambda_\mu}{R^2}\int_0^R \mathrm{d}r\, J_1(r\xi_\beta)J_0(r\xi_\mu)$$
$$\times\left(\Lambda_\mu+\frac{1}{R^2}\sum_{\lambda=1}^{\infty}\frac{\xi_\lambda\Lambda_\lambda}{\kappa^2-\xi_\lambda^2}\int_0^R \mathrm{d}r\, J_1(r\xi_\lambda)J_0(r\xi_\mu)\right)^{-1}.$$

Expressions (8) for the current density components in ξ's space define the conductivity tensor components $\sigma_{ij}^{\mu\mu}(\omega, \xi_\mu, k_z)$ and $\sigma_{ij}^{\mu\beta}(\omega, \xi_\mu, \xi_\beta, k_z)$. Obviously the real part of σ_{ij} (the energy dissipation) should be specified by the poles of $\alpha_{\mu\mu}$ (or $\alpha_{\mu\beta}$). The analysis of expressions (A.6), taking into account (A.2) and (A.5), shows that the dissipation (Re σ_{ij}) is determined by the poles $\kappa^2 = \xi_\mu^2$, ie by the first part of the coefficients $\alpha_{\mu\mu}$:

$$\alpha'_{\mu\mu} = (\mathrm{i}\kappa\delta_1+\delta_2)\frac{\xi_\mu}{\kappa^2-\xi_\mu^2}.$$

Therefore, the second part of $\alpha_{\mu\mu}(\alpha''_{\mu\mu} = \alpha_{\mu\mu}-\alpha'_{\mu\mu})$ and the coefficients $\alpha_{\mu\beta}$, which contain the integral $\int_0^R \mathrm{d}r\, J_1(r\xi_\lambda)J_0(r\xi_\mu)$, have no contributions to Re σ_{ij}. Hence, the collisionless dissipation due to the thermal motion of the plasma particles and to the boundary conditions does not depend on the occurrence of integral (A.3).

In order to estimate the contributions of $\alpha''_{\mu\mu}$ and $\alpha_{\mu\beta}$ to the wave spectrum

$$\mathrm{Re}\,\mathscr{E}^l(\omega, k_z) = 0$$

let us return to (A.4) and differentiate it with respect to ξ_μ. This yields the following differential equation for $A^-(\xi)$

$$\frac{\kappa^2-\xi^2}{\xi}\frac{\mathrm{d}A^-(\xi)}{\mathrm{d}\xi}-\left(1+\frac{\kappa^2+\xi^2}{\xi^2}\right)A^-(\xi) = (\mathrm{i}\kappa\delta_1+\delta_2)\frac{\mathrm{d}\phi(\xi)}{\mathrm{d}\xi} \tag{A.7}$$

where the unity in the bracket in front of $A^-(\xi)$ represents the contribution of integral (A.3). Let the solution of (A.7) be $A^-(\xi) = f[\phi(\xi)]$. One may show that the behaviour of the solution depends on the value of the parameter κ. The dependence on the term $A^-(\xi)$ (the unity in the bracket), corresponding to (A.3), is inessential in the cases when: (*a*) $\kappa \gg \xi$ and (*b*) $\kappa < \xi$ provided that $\mathrm{i}\kappa\delta_1+\delta_2 \simeq 0$. The first case is not interesting as it is dominant in the calculations of the hydrodynamical effects only. In the second case the conditions $\kappa < \xi$ and $\mathrm{i}\kappa\delta_1+\delta_2 \simeq 0$ can be fulfilled if the inequalities

$$\omega \gg \max\left(\frac{v_{T_e}}{R}, k_z v_{T_e}\right) \tag{A.8}$$

or

$$\omega \simeq 0 \tag{A.9}$$

are satisfied. Inequality (A.8) means a weak spatial dispersion of the waves propagating in bounded plasmas, and (A.9) expresses the shielding effect of the plasma particles.

Hence, the omission of integral (A.3) upon obtaining expressions (10) is reasonable only when we investigate the wave spectra in the frequency ranges $\omega \gg k_z v_{T_e}$; $k_z v_{T_i} \ll \omega \ll k_z v_{T_e}$ or $\omega \ll k_z v_{T_\alpha}$. A comparison of the spectra both for high-frequency (Diament *et al* 1966) and for ion-acoustic (Klevans and Mitchell 1970) surface waves with the spectra derived from the equation $D_0 = 0$ shows that the contribution of integral (A.3) to the wave spectra is really inessential.

References

Aliev Yu M and Ferlengi E 1969 *Zh. Eksp. Teor. Fiz.* **57** 1623–8

Aliev Yu M, Gradov O M and Kirii A Yu 1972a *Zh. Eksp. Teor. Fiz.* **63** 112–20

Aliev Yu M and Silin V P 1965 *Zh. Eksp. Teor. Fiz.* **48** 901–12

Aliew Ju M, Gradow O M and Sünder D 1972b *Beitr. Plasma Phys.* **12** 361–72

Andreev N E, Kirii A Yu and Silin V P 1969 *Zh. Eksp. Teor. Fiz.* **57** 1028–39

Bogdankevich L S, Milič B and Rukhadze A A 1967 *Zh. Tekh. Fiz.* **37** 1936–9

Cheng C C and Harris E G 1969 *Phys. Fluids* **12** 1262–70

Chu T K, Bernabei S and Motley R W 1973 *Phys. Rev. Lett.* **31** 211–4

Dawson J and Oberman C 1962 *Phys. Fluids* **5** 517–24

Diament P, Granatstein V L and Schlesinger S P 1966 *J. Appl. Phys.* **37** 1771–8

DuBois D F 1973 *Survey Lectures Int. Congr. on Waves and Instabilities in Plasmas, Innsbruck* (Innsbruck: University of Innsbruck)

DuBois D F and Goldman M V 1967 *Phys. Rev.* **164** 207–22

Franklin R N, Hamberger S M, Lampis G and Smith G J 1971 *Phys. Rev. Lett.* **27** 119–23

Ginzburg V L and Rukhadze A A 1970 *Volny v Magnitoaktivnoi Plazme* (*Waves in a Magnetoactive Plasma*) (Moscow: Nauka)

Goldman R 1969 *Phys. Fluids* **12** 1251–9

Kaw P K and Dawson J M 1971 *Phys. Fluids* **14** 792–4

Klevans E H and Mitchell J D 1970 *Phys. Fluids* **13** 2721–4

Kondratenko A N 1972 *Zh. Tekh. Fiz.* **42** 743–5

Lee K, Kaw P K and Kennel C F 1972 *J. Geophys. Res.* **77** 4197–208

Markeev B M 1971 *Zh. Tekh. Fiz.* **41** 259–65

Moisan M 1971 *Thesis* University of Paris-South (Orsay) No A839

Nishikawa K 1968 *J. Phys. Soc. Jap.* **24** 916–22, 1152–8

Perkins F W and Flick J 1971 *Phys. Fluids* **14** 2012–8

Petiau G 1955 *La Théorie des Fonctions des Bessel, Exposée en Vue de ses Applications à la Physique Mathematique* (Paris: Centre National de la Recherche Scientifique)

Sanmartin J 1970 *Phys. Fluids* **13** 1533–42

Silin V P 1968 *A Survey of Phenomena in Ionized Gases* (Vienna: IAEA) pp 205–37

Tamor S 1973 *Phys. Fluids* **16** 1169–71

Trievelpiece A V and Gould R W 1959 *J. Appl. Phys.* **30** 1784–93

Valeo E and Oberman C R 1973 *Phys. Rev. Lett.* **30** 1035–8

Zhelyazkov I and Nenovski P 1973 *Proc. 11th Int. Conf. on Phenomena in Ionized Gases, Prague* (Prague: Czechoslovak Academy of Sciences, Institute of Physics) p 334